Loop Quantum Cosmology III:
Wheeler–DeWitt Operators

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Abstract

In the framework of loop quantum cosmology anomaly free quantizations of the Hamiltonian constraint for Bianchi class A, locally rotationally symmetric and isotropic models are given. Basic ideas of the construction in (non-symmetric) loop quantum gravity can be used, but there are also further inputs because the special structure of symmetric models has to be respected by operators. In particular, the basic building blocks of the homogeneous models are point holonomies rather than holonomies necessitating a new regularization procedure. In this respect, our construction is applicable also for other (non-homogeneous) symmetric models, e.g. the spherically symmetric one.

1 Introduction

Loop quantum gravity [1], a program for the canonical quantization of general relativity, has lead to a well-understood theory of quantum geometry [2] predicting discrete spectra for geometrical operators like area or volume [3, 4, 5, 6, 7, 8]. The main open problems concern the semiclassical limit and the dynamics of the quantum theory. Recently, a strategy to attack the first problem has been proposed [9] by constructing coherent states which are peaked around a given classical configuration.

As for dynamics, there are a number of anomaly free quantizations of the Hamiltonian constraint [10, 11] and a class of very special solutions [12]. It is, however, not clear how to interpret any possible solution, in part because of conceptual problems, most importantly the problem of time, which are already present in the classical canonical formulation. These classical conceptual questions have been mainly investigated in reduced models subject to a symmetry condition such that one has a good control over all solutions. Most prominent in this context are cosmological models which are homogeneous in space and present examples of mini-superspaces which after reduction can be quantized similar to a conventional

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mechanical system [13, 14]. In the first part of this series [15] a way was proposed how to perform such a symmetry reduction at the kinematical level of loop quantum gravity based on the general concept of symmetric states in diffeomorphism invariant quantum theories of connections [16]. Equivalently, this can be seen as a loop quantization of a mini-superspace, but the quantum states can always be interpreted as generalized, symmetric states of loop quantum gravity. Based on the kinematical level, the next step is to quantize the reduced Hamiltonian constraint of a given model resulting in the Wheeler–DeWitt operator which governs its dynamics. Having such a quantization one can then look whether the classical conceptions regarding the problem of time apply and what they imply for the quantum dynamics [17, 18].

Quantizations of the Hamiltonian constraints for some models are the subject of the present paper. Because the aim is to understand the dynamics of loop quantum gravity, we will follow the basic construction in Ref. [10] as close as possible. Although the ensuing Wheeler–DeWitt operators will be quite similar to the one of the full theory, there is additional input needed in their quantization, essentially due to the following basic ingredient: in order to obtain the curvature components present in the classical constraints, one has to lay a loop in the space manifold whose holonomy approximates the curvature of the Ashtekar connection. In the full theory, one then couples the classical limit with a continuum limit in which all these infinitesimal loops exactly reproduce the curvature components. Moreover, one has to use diffeomorphism invariance in order to ensure that the continuum limit does not affect the quantum constraint. In homogeneous models we then have to face the following problems: First, there is no place for such loops because the reduced models are formulated in a single point and their quantum states are based on point holonomies rather than ordinary holonomies. Therefore, a new regularization is needed which also provides a substitute for the continuum limit which played an important role in the quantization of the Hamiltonian constraint in the full theory. We will do this by extending the point, in which the reduced model is formulated, to an auxiliary manifold using the structure of the symmetry group. This step has already been done in the construction of derivative operators [15]. The next problem then is to lay the loops in such a way that the Wheeler–DeWitt operator respects the symmetry, resulting in the main difference to the operator in the full theory [10].

We will start by reviewing the structure of loop quantum cosmological models in Section 2 and the construction of the Wheeler–DeWitt operator in the full theory in Section 3. Then we will be ready to quantize the Hamiltonian constraint of Bianchi class A models (Section 4) and subsequently of locally rotationally symmetric and isotropic models (Section 5).

2   Bianchi class A, LRS and Isotropic Models

In this section we recall the results of a symmetry reduction for cosmological models and their quantization at the kinematical level. For details we refer to Ref. [15].

Bianchi models are homogeneous models which have a symmetry group S with structure constants c_{IJ}^{K} acting freely and transitively on the space manifold Σ. Bianchi class A

models are characterized by $c^I_{JI} = 0$. Some of them can be reduced by requiring a non-trivial isotropy group which is isomorphic either to $U(1)$ (locally rotationally symmetric, LRS models) or to $SO(3)$ (isotropic models). In all cases, S is a semidirect product of a translation group N and an isotropy group F. We are interested in the reduction of general relativity formulated in real Ashtekar variables and, therefore, need the general expressions of invariant $SU(2)$-connections and $\mathcal{L}SU(2)$-valued density weighted dreibein fields.

An invariant connection can always be decomposed as $A = \phi^i_I \omega^I \tau_i$ using the generators $\tau_j = -\frac{i}{2}\sigma_j$ (with Pauli matrices σ_j) of $G = SU(2)$ and left invariant one-forms ω^I on the translation group N (which is identical to S for Bianchi models). Invariant connections are parameterized by the components ϕ^i_I of a linear map $\phi: \mathcal{L}N \to \mathcal{L}G$ which is arbitrary for general Bianchi models, but restricted to be of the form

$$\phi^i_1 = 2^{-\frac{1}{2}}(a\Lambda^i_1 + b\Lambda^i_2) \quad , \quad \phi^i_2 = 2^{-\frac{1}{2}}(-b\Lambda^i_1 + a\Lambda^i_2) \quad , \quad \phi^i_3 = c\Lambda^i_3 \tag{1}$$

for LRS models and $\phi^i_I = c\Lambda^i_I$ for isotropic models. Λ^i_I is a dreibein which is rotated by the internal $SU(2)$-gauge transformations.

Similarly, an invariant density weighted dreibein field can be decomposed as $E^a_i = \sqrt{g_0}\, p^I_i X^a_I$ with left invariant vector fields on N satisfying $\omega^I(X_J) = \delta^I_J$ and the determinant g_0 of a left invariant metric on N defined by $\omega^1 \wedge \omega^2 \wedge \omega^3 = \sqrt{g_0}\, \mathrm{d}^3x$. Again, for Bianchi models the coefficients p^I_i are arbitrary, but restricted to be of the form

$$p^1_i = 2^{-\frac{1}{2}}(p_a\Lambda^i_1 + p_b\Lambda^i_2) \quad , \quad p^2_i = 2^{-\frac{1}{2}}(-p_b\Lambda^i_1 + p_a\Lambda^i_2) \quad , \quad p^3_i = p_c\Lambda^i_3$$

for LRS models and $p^I_i = p\Lambda^I_i$ for isotropic models.

The symplectic structure is given by

$$\{\phi^i_I, p^J_j\} = \kappa\iota'\delta^i_j\delta^J_I$$

for Bianchi models,

$$\{a, p_a\} = \{b, p_b\} = \{c, p_c\} = \kappa\iota'$$

and vanishing in all other cases for LRS models, and $\{c, p\} = \kappa\iota'$ for isotropic models. Here, $\kappa = 8\pi G$ is the gravitational constant and $\iota' := \iota V_0^{-1}$ the reduced Immirzi parameter with $V_0 := \int_\Sigma \mathrm{d}^3x\sqrt{g_0}$.

Reduced expressions can be derived by inserting invariant connections and dreibein fields into the unreduced expressions of the full theory. In this paper we need the Euclidean parts of the reduced Hamiltonian constraints, which are

$$
\begin{aligned}
g_0^{-1}\mathcal{H}^{(\mathrm{E})} &= g_0^{-1}\epsilon_{ijk}F^i_{IJ}E^I_j E^J_k = -\epsilon_{ijk}c^K_{IJ}\phi^i_K p^I_j p^J_k + \epsilon_{ijk}\epsilon_{ilm}\phi^l_I\phi^m_J p^I_j p^J_k \\
&= -\epsilon_{ijk}c^K_{IJ}\phi^i_K p^I_j p^J_k + \phi^j_I\phi^k_J p^I_j p^J_k - \phi^k_I\phi^j_J p^I_j p^J_k
\end{aligned}
\tag{2}
$$

for Bianchi models,

$$
\begin{aligned}
g_0^{-1}\mathcal{H}^{(\mathrm{E})} &= -(n^{(1)} + n^{(2)})(ap_a + bp_b)p_c - n^{(3)}c(p_a^2 + p_b^2) \\
&\quad + (ap_a + bp_b + cp_c)^2 - \tfrac{1}{2}(ap_a + bp_b)^2 - (cp_c)^2 + \tfrac{1}{2}(ap_b - bp_a)^2
\end{aligned}
\tag{3}
$$

for LRS models, where the constants $n^{(I)}$ specify the Bianchi type, and

$$g_0^{-1}\mathcal{H}^{(\mathrm{E})} = -2(n^{(1)} + n^{(2)} + n^{(3)})cp^2 + 6(cp)^2 \tag{4}$$

for isotropic models, where $n^{(1)} = n^{(2)} = n^{(3)} = 0$ for the isotropic flat model (isotropic Bianchi I), and $n^{(1)} = n^{(2)} = n^{(3)} = 1$ for the isotropic closed model (isotropic Bianchi IX).

To check the constraint algebras, we will also need the diffeomorphism constraints, which are

$$\mathcal{D}_I = -(\kappa\iota')^{-1}c_{IJ}^K\phi_K^i p_i^J \tag{5}$$

for Bianchi models, and vanish identically for LRS models (provided that the Gauß constraint is solved) and isotropic models. So only for Bianchi models demanding an anomaly free representation of the constraint algebra will put restrictions on the quantization of the constraints, although not for all types (for Bianchi type I the structure constants vanish, and so the diffeomorphism constraint).

For a non-vanishing diffeomorphism constraint, the constraint algebra can be derived from the full theory: the Poisson bracket of a diffeomorphism constraint smeared with a shift vector field N^a and a Hamiltonian constraint smeared with lapse function N yields a Hamiltonian constraint with a lapse function given by the shift vector field and the derivatives of the lapse function N. In a reduction to homogeneous configurations, the lapse function is a constant, and so its derivatives vanish. Therefore, the diffeomorphism constraint and the Hamiltonian constraint Poisson commute. Moreover, as can easily be checked by a direct calculation using the Jacobi identity for the structure constants c_{IJ}^K, the diffeomorphism constraint (5) commutes with the Euclidean part of the Hamiltonian constraint (2).

We now come to a description of the kinematical sector of loop quantum cosmology [15]. For Bianchi models, the three scalars $\phi_I^i\tau_i$ are exponentiated to point holonomies [19] taking values in $SU(2)$. The auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}} = L^2(SU(2)^3, \mathrm{d}\mu_{\mathrm{H}}^3)$ consists of functions of these point holonomies being square integrable with respect to the three-fold copy of Haar measure. A basis is given by spin network states which combinatorially can be associated to spin networks with three closed edges meeting in a single 6-vertex x_0. After extending x_0 to an auxiliary manifold, they can be identified with spin networks embedded in this manifold. The diffeomorphism constraint generates transformations which correspond to conjugations in S, e.g. rotations for the Bianchi type IX model.

The auxiliary Hilbert spaces of LRS and isotropic models can be obtained from that of Bianchi models by a further reduction which takes care of the fact that for those models point holonomies which are rotated by an element of the isotropy group are gauge equivalent: while kinematical quantum states for Bianchi models are given by functions $f(h_1, h_2, h_3)$ with $h_I := \exp(\phi_I^i\tau_i) \in SU(2)$, for LRS models quantum states are given by functions

$$f(h_1, \exp(\tfrac{\pi}{2}\Lambda_3)h_1\exp(-\tfrac{\pi}{2}\Lambda_3), h_3)$$

where h_3 is a function of $\Lambda_3 := \Lambda_3^i\tau_i$ via $h_3 = \exp(c\Lambda_3)$, and for isotropic models by functions

$$f(\exp(\tfrac{\pi}{2}\Lambda_2)h_3\exp(-\tfrac{\pi}{2}\Lambda_2), \exp(-\tfrac{\pi}{2}\Lambda_1)h_3\exp(\tfrac{\pi}{2}\Lambda_1), h_3)\,.$$

Therefore, their states can be represented with a reduced number of edges. However, the configuration spaces are not subgroups of $SU(2)^3$, but only subsets given by a union of conjugacy classes. This implies that quantum states can no longer be represented as ordinary spin network states, but as generalized spin network states with insertions (see Ref. [20] for a discussion).

3 The Hamiltonian Constraint in the Full Theory

For real Ashtekar variables, the Hamiltonian constraint in Lorentzian signature is given by [21, 10]

$$\mathcal{H}[N] = \int_\Sigma \mathrm{d}^3 x N (\det q)^{-\frac{1}{2}} \epsilon^{ij}{}_k E_i^a E_j^b \left(F_{ab}^k - 2(1 + \iota^2) E_{[i}^a E_{j]}^b K_a^i K_b^j \right) \tag{6}$$

where F_{ab} are the curvature components of the Ashtekar connection and

$$K_a^i := (\det q)^{-\frac{1}{2}} K_{ab} E^{bi}$$

are the coefficients of the extrinsic curvature of Σ. The extrinsic curvature coefficients are complicated functions of the phase space variables, but luckily they are contained only in the second term of the constraint (6). An important step towards a quantization of the Hamiltonian constraint has been done in Refs. [10, 23] leading to a first consistent regularization of the constraint. An important input was a procedure to treat the second term of the constraint containing the curvature coefficients K_a^i, which has been achieved by first writing

$$\mathcal{H} = 2(\det q)^{-\frac{1}{2}}(1 + \iota^2)\operatorname{tr}([K_a, K_b][E^a, E^b]) - \mathcal{H}^{(\mathrm{E})} \tag{7}$$

where

$$\mathcal{H}^{(\mathrm{E})} = 2(\det q)^{-\frac{1}{2}}\operatorname{tr}(F_{ab}[E^a, E^b]) \tag{8}$$

is the Euclidean constraint. In these formulae the variables are written as $\mathcal{L}SU(2)$-valued, e.g. $E^a := E_i^a \tau^i$. The square brackets and tr denote the commutator and trace in this Lie algebra.

It is then necessary to represent the curvature components $K_a = K_a^i \tau_i$ as quantum operators. The recipe of Ref. [10] starts from the observation that the trace

$$K := \int_\Sigma \mathrm{d}^3 x \sqrt{\det q} K_{ab} q^{ab} = \int_\Sigma \mathrm{d}^3 x K_a^i E_i^a$$

is the time derivative of the volume (independent of the space-time signature) which can in a Hamiltonian formulation be written as Poisson bracket of Hamiltonian and volume:

$$K = \mathcal{L}_t V = -\{V, \mathcal{H}^{(\mathrm{E})}\}.$$

When one then uses the important fact that $\{\Gamma_a^i, K\} = 0$ [24] to represent the components of the intrinsic curvature as

$$K_a^i = \frac{\delta K}{\delta E_i^a} = (\kappa\iota)^{-1}\{A_a^i, K\},$$

translates all Poisson brackets into commutators, expresses the connection coefficients by holonomies and uses a quantization of the volume [3, 6, 7] one can trace back the quantization of the second term of (6) to a quantization of the first term.

Furthermore, in Ref. [10] a quantization of the Euclidean constraint $\mathcal{H}^{(\mathrm{E})}$ has been given which completes the quantization of $\mathcal{H}$. Here a key point is that in a diffeomorphism invariant context only expressions with the correct density weight can be quantized without a background structure. This means that the determinant of the metric must not be absorbed into the lapse function in order to have a polynomial constraint because this would change the density weight. Instead, the key identity

$$(\det q)^{-\frac{1}{2}}[E^a, E^b]^i = \epsilon^{abc} e_c^i = 2\epsilon^{abc} \frac{\delta V}{\delta E_i^c} = 2(\kappa\iota)^{-1} \epsilon^{abc}\{A_c^i, V\} \qquad (9)$$

is used and quantized by turning the Poisson bracket into a commutator, representing the connection coefficients by a holonomy and using the known volume operator $\hat{V}$.

To regularize the constraint, the manifold Σ is triangulated in a way adapted to the graph of a cylindrical function on which the constraint operator is to act. This is done in such a manner that for each triple of edges meeting in a vertex v a triangle α_{ij}, which consists of two pieces s_i and s_j of two edges and a third curve a_{ij} connecting their endpoints, is chosen as a base of the tetrahedron spanned by s_i, s_j and a piece s_k of the third edge. The rest of Σ is triangulated arbitrarily. Using the above identities and an expansion of holonomies for infinitesimal length one can see that the vertex contributions

$$\epsilon^{ijk} \operatorname{tr}(h_{\alpha_{ij}} h_{s_k}[h_{s_k}^{-1}, \hat{V}])$$

summed over all triples of edges meeting in v have the correct classical limit for a local contribution of the Euclidean constraint.

Contained in the classical limit is a continuum limit in which the width of the triangulation vanishes. In the quantum theory, however, such a continuum limit is trivial provided we quantize the constraint on the Hilbert space $\mathcal{H}_{\mathrm{diff}}$ where the diffeomorphism constraint is solved. In this case two operators which are obtained by a triangulation and a refinement of it are identical because their actions on a fixed cylindrical function by construction differ only by a diffeomorphism moving the edges a_{ij}.

When acting on a cylindrical function the Euclidean constraint changes the graph due to the holonomies to the curves a_{ij} in such a way that in the neighborhood of any non-planar vertex for each two edges incident there a new edge connecting them is generated. The action of the operator is well understood, but there are only some rather trivial explicitly known solutions and the complete solution space is not under any control. Furthermore, there are a lot of ambiguities in the construction of a particular operator and it is not clear which one to use. This question can probably be answered only by investigating whether the theory has the correct classical limit. A strategy proposed here is to use similar regularizations of the Hamiltonian constraint on appropriate sectors of symmetric states for a comparison with the corresponding classical mini-superspaces.

4   Hamiltonian Constraints for Bianchi Models

We are now ready to present a quantization of the Hamiltonian constraints for cosmological models, starting with Bianchi models. Our construction will follow that of Ref. [10] in the full theory, but there are important points where new input is necessary. This is the case because, first, our regularization has to respect the symmetry conditions and, second, because connection coefficients are coded in point holonomies rather than ordinary holonomies. Furthermore, Eq. (2) shows that the constraint contains two different parts: a part linear in the scalar fields which via the structure constants c_{IJ}^K depends on the Bianchi type, and a second part quadratic in the scalar fields which is independent of the type. Therefore, although the original expression (6) does not bear any reference to the symmetry type, all the reduced models will have different Wheeler–DeWitt operators and different dynamics.

As already mentioned, we will follow the procedure presented in Ref. [10] and recalled in Section 3 when quantizing the expressions (2), (3) and (4) on the respective kinematical Hilbert spaces. Some steps can immediately be copied, for instance we can restrict ourselves to the Euclidean constraints because the additional part in the Lorentzian constraint can be treated in complete analogy to the full theory by using the extrinsic curvature K. Furthermore, we will also make use of commutators of holonomies with the volume operator, i.e. this operator will again play a prominent role. At this place it is fortunate that it simplifies in symmetric regimes [20] such that an analysis of the matrix elements of the Hamiltonian constraint will be easier.

What we cannot copy in our reduced models is the use of a triangulation of space and the subsequent continuum limit. At first sight, we may seem to be in a better situation because we are considering a finite dimensional model and do not have to bother about a continuum limit. But in the regularization used in the full theory this limit also serves the purpose to express the curvature components F_{ab}^i entering (6) in terms of closed loops in the following way: As recalled above, one starts with a triangulation of space which is adapted to the graph of a spin network the Hamiltonian constraint operator is supposed to act on. Forming a triangular loop α_{ij} based in v one can compute the holonomy $h_{\alpha_{ij}}$ appearing in each vertex contribution discussed in Section 3. In the continuum limit any such loop shrinks to v such that the holonomies approximate the curvature components. Here one makes use of the expansion

$$h(A) = 1 + \tfrac{1}{2}\epsilon^2 u^a v^b F_{ab}^i \tau_i + O(\epsilon^3)$$

for a holonomy h along an infinitesimal parallelogram spanned by two edges of length ϵ with directions u^a and v^a in Euclidean space. This is the final ingredient needed to turn all contributions to the classical Hamiltonian constraint into quantum operators.

The ensuing Wheeler–DeWitt operator then contains the holonomies along all the triangular loops α_{ij} as multiplication operators creating new edges a_{ij}. Although the construction of these loops depends on the triangulation, the action of the operator is triangulation independent once it is considered to act on the kinematical Hilbert space where the diffeomorphism constraint is solved. In this sense, the continuum limit is trivial in the quantum

theory because for a finer triangulation the newly created edges are just moved using a diffeomorphism thus representing the same diffeomorphism invariant quantum state.

Obviously, we cannot use these techniques to generate the curvature components in a symmetric model: First, the prescription to generate new edges violates our symmetry conditions (e.g., in the Bianchi models there are just the three edges obtained by smearing the point holonomies, and no edge connecting two of them can be generated). Second, we do not have a continuum limit in which course loops would shrink and could be approximated using curvature components.

We will present now alternative techniques which are adapted to the symmetry and which make use of the methods developed to deal with point holonomies. As in the computation of the derivative operators in Ref. [15] the regularizing edges will be important. All this will be discussed in detail for the Bianchi models and then used in LRS and isotropic models.

As demonstrated in Ref. [10], we have to provide the Euclidean part (2) of the Hamiltonian constraint with the correct density weight in order to be able to quantize it in a background independent manner. We thus divide the earlier expression by $\sqrt{\det q} = \sqrt{\frac{1}{6}g_0|\epsilon^{ijk}\epsilon_{IJK}p_i^I p_j^J p_k^K|}$, multiply with a constant (due to homogeneity) lapse function N and integrate over the manifold Σ:

$$\mathcal{H}^{(\mathrm{E})}[N] = -\int_\Sigma \mathrm{d}^3 x \, N (\det q)^{-\frac{1}{2}} g_0 \epsilon^{ijk} p_i^I p_j^J F_{IJk}$$

with

$$F_{IJ}^i = -c_{IJ}^K \phi_K^i + \epsilon^i{}_{jk}\phi_I^j \phi_J^k \, .$$

This can immediately be integrated to

$$
\begin{aligned}
\mathcal{H}^{(\mathrm{E})}[N] &= -V_0 N \left(\frac{1}{6}\left|\epsilon^{lmn}\epsilon_{LMN}p_l^L p_m^M p_n^N\right|\right)^{-\frac{1}{2}} \epsilon^{ijk} p_i^I p_j^J F_{IJk} \\
&= -\frac{1}{6}V_0(\kappa\iota')^{-1} N \left(\frac{1}{6}\left|\epsilon^{lmn}\epsilon_{LMN}p_l^L p_m^M p_n^N\right|\right)^{-\frac{1}{2}} \left\{\phi_K^k, \epsilon_{OPQ}\epsilon^{ijl}p_i^O p_j^P p_l^Q\right\} \epsilon^{IJK} F_{IJk} \\
&= -2(\kappa\iota')^{-1} N \epsilon^{IJK} \left\{\phi_K^k, V\right\} F_{IJk}
\end{aligned}
$$

where we used

$$V = V_0 \sqrt{\frac{1}{6}|\epsilon^{ijk}\epsilon_{IJK}p_i^I p_j^J p_k^K|}$$

and

$$\left\{\phi_K^k, \epsilon_{MNL}\epsilon^{ijl}p_i^M p_j^N p_l^L\right\} = 3\kappa\iota' \epsilon^{ijk}\epsilon_{MNK}p_i^M p_j^N \tag{10}$$

which, divided by $\sqrt{\det q}$ is essentially the key identity (9) used in Ref. [10] to quantize the inverted triad components. The expression of F_{IJ}^k in terms of the scalar fields depends on the particular symmetry group, i.e. on the Bianchi type, which will be taken care of when we choose the route of loops below.

4.1 Auxiliary Manifolds

But before this we have to discuss our regularization scheme which, as already noted, has to be different from that in the full theory because we cannot use a triangulation as regulator and the continuum limit in connection with diffeomorphism invariance to remove it. After integrating over Σ we arrived above at an expression for the Hamiltonian constraint which is completely composed of the reduced field components sitting in the single point x_0 of the reduced manifold $B := \Sigma/S = \{x_0\}$. Obviously in this reduced description there is no substitute for a triangulation of space which could be used as regulator. Note, however, that in order to compute derivative operators in point holonomies [15] we already had to smear point holonomies along auxiliary edges. To that end we have, in the present context of cosmological models, introduced an auxiliary manifold $\overline{S/F}$ (a suitable compactification of the homogeneous space S/F) in which these edges are to lie, where in the construction of both the auxiliary manifold and the edges we made use of the structure of $\overline{S/F}$ as a homogeneous space. This structure can now also be used in order to regulate the Hamiltonian constraint, but again we cannot simply copy the procedure of the full theory (applied to the 6-vertex lying in the auxiliary manifold) because this would spoil the symmetry (we are not allowed to create new edges, but only to retrace existing ones).

Although we then manage to respect the symmetry, we immediately have to face another problem: In order to approximate the curvature component by the attached loops and to recover the correct classical limit we have to shrink the loops to the vertex, which in the full theory was achieved by the continuum limit which we now do not have at our disposal. Our only possibility is to shrink the whole auxiliary manifold and with it the auxiliary edges to the point x_0. In this limit we will recover the classical limit, whereas the quantum theory is independent of the extension of our auxiliary manifold. This is completely analogous to the full theory, where the diffeomorphism invariant quantum theory is triangulation independent.

Let us now follow in detail the program outlined above. We have a compact homogeneous auxiliary manifold $\overline{S/F}$ containing the point x_0. Each point of $\overline{S/F}$ can be reached from x_0 by following pieces of integral curves to the left invariant vector fields X_I (which also have been used to define the auxiliary edges smearing point holonomies). If we always normalize the invariant vector fields in such a way that their closed (by construction of $\overline{S/F}$) integral curves have a fixed parameter length, then shrinking the auxiliary manifold is equivalent to multiplying the vector field by a number ϵ smaller than one. So we can describe the shrinking of $\overline{S/F}$ to $\overline{S/F}_\epsilon$ by replacing X_I with ϵX_I where eventually we will consider the limit $\epsilon \to 0$ in connection with the classical limit.

We now illustrate the shrinking procedure for the Bianchi I model on the manifold $\mathbb{R}^3$. In terms of coordinates x^I adapted to the symmetry (the usual Cartesian coordinates), left invariant vector fields are $X_I = \frac{\partial}{\partial x^I}$. The manifold $\mathbb{R}^3$ is non-compact, but we can compactify it to a three-torus, which then plays the role of $\overline{S/F}$, by restricting the coordinates to $0 \leq x^I \leq 1$ and identifying the points with $x^I = 0$ and $x^I = 1$ for each I. By this compactification the integral curves of the vector fields X_I are rendered closed and will

be used as auxiliary edges based in a base point x_0 in the three-torus. Denoting the flow generated by a vector field X by $\Phi_t(X)\colon \overline{S/F} \to \overline{S/F}$, we can write

$$\overline{S/F} = \left\{ \Phi_{t_3}(X_3)\Phi_{t_2}(X_2)\Phi_{t_1}(X_1)x_0 : t_I \in S^1 \cong \mathbb{R}/\mathbb{Z} \right\}.$$

The shrinking manifolds $\overline{S/F}_\epsilon$ are then obtained for fixed t_I-intervals by replacing X_I with ϵX_I everywhere. Then the three-torus as well as the closed integral curves shrink and, for infinitesimal ϵ, holonomies along the integral curves (i.e. the smeared point holonomies) are approximated by ϵ times the respective scalar field component ($h_I = 1 + \epsilon\phi_I + O(\epsilon^2)$), and volume integrals over the auxiliary manifold are approximated by ϵ^3 times the reduced quantity defined in the base point.

4.2 Regularization

Our previous expression for the Hamiltonian constraint was written down in the reduced description on the point x_0. To extend it to the auxiliary manifold we just have to integrate it, yielding just a factor of ϵ^3 because the constraint is constant on $\overline{S/F}_\epsilon$. We thus have

$$\mathcal{H}_\epsilon^{(\mathrm{E})}[N] := \int_{\overline{S/F}_\epsilon} \mathrm{d}^3x\, \mathcal{H}^{(\mathrm{E})}[N] = -2\epsilon^3(\kappa\iota')^{-1}N\epsilon^{IJK}\left\{\phi_K^k, V\right\}F_{IJk} \qquad (11)$$

as point of departure for the quantization. An integration is mandatory here in order to obtain an expression without density weight (which can be defined using scale transformations respecting the symmetry). The original, unregulated expression can be recovered as (in this expression the limit is trivial which will, however, not be the case if we introduce holonomy variables below)

$$\lim_{\epsilon\to 0} \epsilon^{-3} \int_{\overline{S/F}_\epsilon} \mathrm{d}^3x\, \mathcal{H}^{(\mathrm{E})}[N] = \mathcal{H}^{(\mathrm{E})}[N]. \qquad (12)$$

One factor of ϵ can be used to express $\epsilon\phi_K^k$ as infinitesimal holonomy, whereas the remaining square of ϵ will be used to express the curvature components via a holonomy along an infinitesimal loop, to which we turn now.

Because the curvature components, expressed in terms of scalar fields, are the only place where the Hamiltonian constraint depends on the symmetry group, we have to make use of its structure when expressing the curvature by means of suitably laid loops. This can be done most easily by composing the loop of pieces of integral curves to the invariant vector fields, i.e. as a geodesic parallelogram in an arbitrary homogeneous metric. E.g., in the Bianchi I model we can use a chart with coordinates x^I which is adapted to the symmetry in such a way that the (commuting) invariant vector fields are given by $X_I = \frac{\partial}{\partial x^I}$. Using the flows $\Phi_t(X_I)$ generated by these vector fields, we can form parallelograms in $\overline{S/F}$ composed of the four edges $c_1\colon t \mapsto \Phi_t(X_I)x_0$, $c_2\colon t \mapsto \Phi_t(X_J)c_1(1)$, $c_3\colon t \mapsto \Phi_{-t}(X_I)c_2(1)$ and $c_4\colon t \mapsto \Phi_{-t}(X_J)c_3(1)$, for all $1 \le I \ne J \le 3$, which is closed because the vector fields X_I and X_J commute. We can now use the constant scalar fields $\phi_I = \phi(X_I)$ parameterizing

a homogeneous connection $A_a^i = \phi_K^i \omega_a^K$ to compute the holonomy along the parallelogram: $\exp(\phi_I) \exp(\phi_J) \exp(-\phi_I) \exp(-\phi_J)$ using, e.g.

$$h_{c_1}(A) = \mathcal{P} \exp \int_{c_1} \mathrm{d}t \, \dot{e}^a A_a^i \tau_i = \mathcal{P} \exp \int_{c_1} \mathrm{d}t \, X_I^a \omega_a^K \phi_K^i \tau_i = \exp(\phi_I^i \tau_i) \, .$$

What remains is to note that point holonomies were defined in just the same way by calculating holonomies along auxiliary edges defined as integral curves to an invariant vector field [15]. We only have to take care that we run through a complete closed integral curve to a single invariant vector field which is possible due to our construction of the compactification $\overline{S/F}$. There are then two possibilities to shrink the parallelogram: First we can fix the homogeneous manifold and only shrink the curves by constraining the parameter t above to an interval $[0, \epsilon]$. This is the usual procedure for a regularization in a three-dimensional theory, but is not applicable here because we can identify a holonomy along an integral curve with a point holonomy only if the curve is closed and based in the point x_0. Thus we have to use the second shrinking procedure where both the auxiliary manifold and the curves embedded into it shrink by using the vector fields ϵX_I as generators of both the auxiliary manifold $\overline{S/F}_\epsilon$ and of their own closed integral curves. Now the holonomies are always along closed integral curves and can be identified with point holonomies for all values of ϵ, but due to $\phi(\epsilon X_I) = \epsilon \phi_I$ (linearity of the map $\phi \colon \mathcal{L}N \to \mathcal{L}G$) we now have the ϵ-dependent holonomy

$$
\begin{aligned}
&\exp(\epsilon \phi_I) \exp(\epsilon \phi_J) \exp(-\epsilon \phi_I) \exp(-\epsilon \phi_J) \\
&= (1 + \epsilon \phi_I + \tfrac{1}{2} \epsilon^2 \phi_I^2)(1 + \epsilon \phi_J + \tfrac{1}{2} \epsilon^2 \phi_J^2)(1 - \epsilon \phi_I + \tfrac{1}{2} \epsilon^2 \phi_I^2)(1 - \epsilon \phi_J + \tfrac{1}{2} \epsilon^2 \phi_J^2) + O(\epsilon^3) \\
&= 1 + \epsilon^2 (\phi_I \phi_J - \phi_J \phi_I) + O(\epsilon^3) \\
&= 1 + \epsilon^2 \epsilon_{ijk} \phi_I^i \phi_J^j \tau^k + O(\epsilon^3) = 1 + \epsilon^2 F_{IJ}^k \tau_k + O(\epsilon^3) \, .
\end{aligned}
$$

Thus we have a product of point holonomies which, for infinitesimal ϵ, reproduces the curvature components for Bianchi I.

For other Bianchi models with a non-Abelian symmetry group, the last equation does not recover the correct curvature components because of the additional term containing the structure constants c_{IJ}^K of the symmetry group. But, also due to the non-Abelian nature of the symmetry group, the parallelogram constructed above will not be closed, even up to the order of ϵ^2 because now the vector fields used to define the edges of the parallelogram do not commute. As is well-known from differential geometry, the endpoint $c_4(1)$ of the above parallelogram constructed from the vector fields ϵX_I and ϵX_J is a distance $[\epsilon X_I, \epsilon X_J]$ away from the origin. So we can close our parallelogram by running through the curve $c_5 \colon t \mapsto \Phi_{-t}([\epsilon X_I, \epsilon X_J]) c_4(1)$ after following the curves c_1 to c_4. The ensuing curve is closed to the order of ϵ^3 and can be closed by connecting the points $c_5(1)$ and x_0 which affects the holonomy along the loop only up to the same order. As holonomy we obtain

$$
\begin{aligned}
&\exp(\epsilon \phi_I) \exp(\epsilon \phi_J) \exp(-\epsilon \phi_I) \exp(-\epsilon \phi_J) \exp(-\epsilon^2 \phi([X_I, X_J])) \\
&= \exp(\epsilon \phi_I) \exp(\epsilon \phi_J) \exp(-\epsilon \phi_I) \exp(-\epsilon \phi_J) \exp(-\epsilon^2 c_{IJ}^K \phi_K)
\end{aligned}
$$

which, thanks to the commutator of left invariant vector fields, depends explicitly on the structure of the symmetry group. Note also that the commutator of two vector fields results in a factor of ϵ^2 for only one power of ϕ in the last exponential. Using the previous calculations we can easily convince ourselves that this leads to the correct curvature components:

$$
\begin{aligned}
&\exp(\epsilon\phi_I)\exp(\epsilon\phi_J)\exp(-\epsilon\phi_I)\exp(-\epsilon\phi_J)\exp(-\phi([\epsilon X_I,\epsilon X_J])) \\
&= (1+\epsilon^2\epsilon_{ijk}\phi_I^i\phi_J^j\tau^k + O(\epsilon^3))(1-\epsilon^2 c_{IJ}^K\phi_K + O(\epsilon^4)) \\
&= 1+\epsilon^2(-c_{IJ}^K\phi_K^k + \epsilon_{ijk}\phi_I^i\phi_J^j)\tau_k + O(\epsilon^3) = 1+\epsilon^2 F_{IJ}^k\tau_k + O(\epsilon^3)\,.
\end{aligned}
$$

We are now in a position to put all ingredients together in order to arrive at our regulated classical expression of the Hamiltonian constraint formulated on the regularizing manifold $\overline{S/F}_\epsilon$. Comparing with Eq. (11), we see that

$$
\begin{aligned}
\mathcal{H}_\epsilon^{(\mathrm{E})}[N] \;\;:=\;\; & 4(\kappa\iota')^{-1}N\sum_{IJK}\epsilon^{IJK}\operatorname{tr}(\exp(\epsilon\phi_I)\exp(\epsilon\phi_J)\exp(-\epsilon\phi_I)\exp(-\epsilon\phi_J) \\
& \times \exp(-\epsilon^2\phi([X_I,X_J]))\{\exp\epsilon\phi_K,V\})
\end{aligned}
$$

has the correct limit (see Eq. (12))

$$
\mathcal{H}^{(\mathrm{E})}[N] = \lim_{\epsilon\to 0}\epsilon^{-3}\mathcal{H}_\epsilon^{(\mathrm{E})}[N]
$$

when the regulator is removed.

4.3 Quantization

From now on we can again follow the steps of Ref. [10] in order to quantize the constraint: exponentials of scalars are replaced by (point) holonomies, the volume is quantized to the operator of Ref. [20] and the Poisson bracket is replaced with $(i\hbar)^{-1}$ times a commutator. The result is

$$
\hat{\mathcal{H}}^{(\mathrm{E})}[N] = -4i(\iota' l_{\mathrm{P}}^2)^{-1}N\sum_{IJK}\epsilon^{IJK}\operatorname{tr}\left(h_I h_J h_I^{-1}h_J^{-1}h_{[I,J]}^{-1}[h_K,\hat{V}]\right) \tag{13}
$$

where h_I is the holonomy along the I-th regularizing edge, interpreted as multiplication operator, and $h_{[I,J]}$ depends on the symmetry group in the following way

$$
h_{[I,J]} := \prod_{K=1}^{3}(h_K)^{c_{IJ}^K}\,.
$$

What remains to do is to use the decomposition (7) and the quantization of the Euclidean part of the constraint in order to quantize the Lorentzian constraint. In the reduced

formulation the decomposition reads (for a constant lapse function N)

$$
\begin{aligned}
\mathcal{H}[N] &= 2(1+\iota^2)\int_\Sigma \mathrm{d}^3x N(\det q)^{-\frac{1}{2}}\operatorname{tr}([K_a,K_b][E^a,E^b]) - \mathcal{H}^{(\mathrm{E})}[N] \\
&= 2(1+\iota^2)\int_\Sigma \mathrm{d}^3x N\sqrt{g_0}\left|\det(p_K^l)\right|^{-\frac{1}{2}}\operatorname{tr}([k_I,k_J][p^I,p^J]) - \mathcal{H}^{(\mathrm{E})}[N] \\
&= -(1+\iota^2)V_0 N\epsilon_{ijk}k_I^i k_J^j\left|\det(p_K^l)\right|^{-\frac{1}{2}}\epsilon^{klm}p_l^I p_m^J - \mathcal{H}^{(\mathrm{E})}[N] \\
&= -2(1+\iota^2)(\kappa\iota')^{-3}V_0^{-2}N\epsilon_{ijk}\epsilon^{IJK}\{\phi_I^i,K\}\{\phi_J^j,K\}\{\phi_K^k,V\} - \mathcal{H}^{(\mathrm{E})}[N]
\end{aligned}
$$

where we defined $K_a^i =: k_I^i\omega_a^I$, $K = \int \mathrm{d}^3x K_a^i E_i^a = V_0 k_I^i p_i^I$ and used Eq. (10). Regularized as above, we obtain

$$
\begin{aligned}
\mathcal{H}_\epsilon[N] &= -2(1+\iota^2)(\kappa\iota)^{-3}V_0 N\epsilon_{ijk}\epsilon^{IJK}\epsilon^3\{\phi_I^i,K\}\{\phi_J^j,K\}\{\phi_K^k,V\} - \mathcal{H}_\epsilon^{(\mathrm{E})}[N] \\
&= 8(1+\iota^2)(\kappa\iota)^{-3}V_0 N\epsilon^{IJK}\operatorname{tr}(\{\exp(\epsilon\phi_I),K\}\{\exp(\epsilon\phi_J),K\}\{\exp(\epsilon\phi_K),V\}) \\
&\quad - \mathcal{H}_\epsilon^{(\mathrm{E})}[N] + O(\epsilon^3)
\end{aligned}
$$

which can again be quantized by replacing the exponentials of scalars by (point) holonomies and the Poisson brackets by $(i\hbar)^{-1}$ times a commutator:

$$
\hat{\mathcal{H}}[N] = 8i(1+\iota^2)(\iota l_{\mathrm{P}}^2)^{-3}V_0 N\epsilon^{IJK}\operatorname{tr}\left([h_I,\hat{K}][h_J,\hat{K}][h_K,\hat{V}]\right) - \hat{\mathcal{H}}^{(\mathrm{E})}[N]. \qquad (14)
$$

As in the full theory, we use here the quantization

$$
\hat{K} = i\hbar^{-1}\left[\hat{V},\hat{\mathcal{H}}^{(\mathrm{E})}[1]\right]
$$

of the integrated extrinsic curvature.

A non-vanishing cosmological constant Λ can be included simply by adding the contribution $2N\Lambda\hat{V}$ using the volume operator of Ref. [20].

Of course, there are factor ordering ambiguities which we ignored in writing the expression above. Furthermore, as written down, the operator is not symmetric which would have to be achieved by choosing an appropriate factor ordering. In the full theory these issues are unsolved and one purpose of the reduced models discussed here can be to gain insights by studying them in a simplified, highly symmetric regime. We will, however, not enter this discussion because it requires a detailed study of these models, whereas here we are mainly interested in general aspects.

It is immediate to see that the operator (14) is manifestly independent of the regulator ϵ which was used to reformulate the classical expression in terms of holonomy variables. This is so because, in contrast to the classical procedure where the scalar fields are fixed in the limit $\epsilon \to 0$ in which the regulator is removed, in the quantum theory the scalars diverge in this point limit in such a way that smeared holonomies are equal to point holonomies independent of ϵ. Recall that the quantum configuration space contains distributional scalars which are needed to accomplish this result. Therefore, it does not matter whether

we formulate the operator on the auxiliary manifold $\overline{S/F}$, on one of the shrunk manifolds $\overline{S/F}_\epsilon$ or even in the point limit using exclusively point holonomies. In each of these formulations we have the same state space and the same action of the Hamiltonian constraint operator. This feature, which here arose after smearing the scalar fields to regulate classical expressions, substitutes the diffeomorphism invariance which in the full theory is used to show that the continuum limit removing the regulator there is trivial in the quantum theory. Thus, although formulated in a conceptually different way, the quantization of the Hamiltonian constraint in the full theory [10] and ours for the Bianchi models share the same key properties. But of course our discussion was technically simplified by the fact that we only have to regard 6-vertices, whereas in the full theory there can be (and have to be taken into account) vertices of arbitrary valence.

It is immediate to see that the constraints are represented in an anomaly free manner: the only non-trivial part is to check that diffeomorphisms commute with the Hamiltonian constraint which can easily be seen after recalling that diffeomorphisms, i.e. inner automorphisms on the auxiliary manifold, just move the three edges of homogeneous spin networks, whereas the Hamiltonian constraint operator fixes them and only changes their labels and the contractor.

5   Hamiltonian Constraints for LRS and Isotropic Models

Similarly to the treatment of volume operators in Ref. [20] we can use the expression (14) and insert the rotated holonomies $h_2 = \exp(\frac{\pi}{2}\Lambda_3)h_1\exp(-\frac{\pi}{2}\Lambda_3)$ to arrive at the constraint operator for LRS models, and $h_1 = \exp(\frac{\pi}{2}\Lambda_2)h_3\exp(-\frac{\pi}{2}\Lambda_2)$, $h_2 = \exp(-\frac{\pi}{2}\Lambda_1)h_3\exp(\frac{\pi}{2}\Lambda_1)$ to arrive at the operator for isotropic models. The resulting operator then only contains the holonomy operators h_1, h_3 for LRS models and h_3 for isotropic models, and in addition operators coming from the exponentials of Λ_I which manipulate the insertions (see Ref. [20]).

By construction, all these operators are gauge invariant, but the intermediate action of holonomies and the volume operator is on gauge non-invariant states. This means that the techniques used for gauge invariant states of isotropic models in Ref. [20] have to be generalized so as to deal with non-invariant states. In particular, the volume operator derived there is not sufficient to calculate the matrix elements of the Hamiltonian constraint for isotropic models.

In contrast to Bianchi models, where the three scalars ϕ_I and so the three point holonomies h_I are independent, for LRS and isotropic models there are restrictions for the point holonomies (see Eq. (1)) leading to the rotated holonomies above. This implies that we can apply the following lemma, which has already been used in Ref. [20], in order to simplify the constraint operator which contains a product of several holonomy operators.

Lemma 1 *Let* $g := \exp(A\tau_i)$ *and* $h := \exp(B\tau_j)$ *with* $A, B \in \mathbb{R}$, $i \neq j$ *be matrices in the*

fundamental representation of $SU(2)$. *Then*

$$gh = hg + h^{-1}g + hg^{-1} - \text{tr}(gh).$$

Let us illustrate this for the flat isotropic model, which is obtained by restricting the Bianchi I model to isotropy. We thus start from the expression (13) with all structure constants vanishing, and insert the rotated holonomies h_1 and h_2 in terms of h_3. Then two different holonomies h_I and h_J, $I \neq J$, appearing in the operator are always of a form such that Lemma 1 applies and we can insert

$$h_J h_I^{-1} = h_I^{-1}h_J + h_I h_J + h_I^{-1}h_J^{-1} - \text{tr}(h_I^{-1}h_J)$$

simplifying the sum in Eq. (13) to

$$\sum_{IJK} \epsilon^{IJK}\left(\text{tr}([h_K,\hat{V}]) + \text{tr}(h_I^2[h_K,\hat{V}]) + \text{tr}(h_J^{-2}[h_K,\hat{V}]) - \text{tr}(h_I^{-1}h_J)\,\text{tr}(h_I h_J^{-1}[h_K,\hat{V}])\right).$$

The first trace is symmetric in (in fact independent of) I and J and therefore cancels in presence of ϵ^{IJK}, and the rest can be written as

$$\sum_{IJK} \epsilon^{IJK}\left(\text{tr}((h_I^2 - h_I^{-2})[h_K,\hat{V}]) - \text{tr}(h_I^{-1}h_J)\,\text{tr}(h_I h_J^{-1}[h_K,\hat{V}])\right).$$

Up to now the explicit expression for the rotated holonomies h_1 and h_2 in terms of h_3 have not been used, but only the fact that Lemma 1 applies because of the special character of rotated holonomies. To arrive at the final expression we have to insert the explicit formulae for h_1, h_2 which leads to a product of h_3-holonomies and exponentials of Λ_1, Λ_2. Here we can again apply Lemma 1 in order to collect all factors of h_3 into a single power of h_3 (except for the one appearing on the right hand side of the volume operator when the commutator is written out) and a holonomy independent factor coming from the exponentials of Λ_I. Alternatively, we can right from the beginning apply Lemma 1 to the definition of h_1, h_2:

$$\begin{aligned}
h_1 &= \exp(\tfrac{\pi}{2}\Lambda_2)h_3\exp(-\tfrac{\pi}{2}\Lambda_2)\\
&= \exp(\tfrac{\pi}{2}\Lambda_2)\left(\exp(-\tfrac{\pi}{2}\Lambda_2)h_3 + \exp(\tfrac{\pi}{2}\Lambda_2)h_3 + \exp(-\tfrac{\pi}{2}\Lambda_2)h_3^{-1} - \text{tr}(\exp(-\tfrac{\pi}{2}\Lambda_2)h_3)\right)\\
&= h_3 + h_3^{-1} + \exp(\pi\Lambda_2)h_3 - \exp(\tfrac{\pi}{2}\Lambda_2)\text{tr}(\exp(-\tfrac{\pi}{2}\Lambda_2)h_3)
\end{aligned}$$

and similarly for h_2 such that h_3 appears always at the right hand side. After some final applications of the lemma we can collect all the factors containing Λ_I in a single operator which manipulates the insertion, whereas the power of h_3 changes the spin of isotropic spin networks.

Once the techniques of isotropic spin networks are generalized to gauge non-invariant states, the calculation of matrix elements of the Hamiltonian constraint operator in isotropic models can be pursued. However, although there is no new input required in addition to

the procedure outlined here, the actual calculation can be expected to be quite cumbersome and we do not present any further details. These computations are needed when one begins to study specific models and are well suited to be done numerically. In LRS models the situation is similar: the essential steps have been outlined, but more work had to be done for specific models.

Because for LRS and isotropic models the diffeomorphism constraint vanishes on gauge invariant states, the constraint algebra is represented anomaly free in a trivial way.

6   Discussion

In this paper we have presented quantizations of Hamiltonian constraints for some cosmological models in the framework of loop quantum cosmology. The strategy was to be as close to the full theory of loop quantum gravity as possible in order to be able to compare with and to draw conclusions for the full theory. We have seen that this is possible to a large extent, but with necessary additional input taking care of the symmetry. In particular, homogeneous models are formulated in a single point x_0 and there is no continuum limit which in combination with diffeomorphism invariance plays an important role in the regularization of the Hamiltonian constraint in the full theory. We have seen that an extension of the point x_0 to an auxiliary manifold, which already appeared in the regularization of derivative operators, provides the additional structure. The Hamiltonian constraint is regularized on the auxiliary manifold, the extension of which serves as regulator, i.e. the classical expression is recovered if the auxiliary manifold is shrunk to the point x_0. But the quantization of the regulated expression is independent of the extension, and so we have an exact analog of the situation in the full theory, where the continuum limit removes the regulator classically and the quantization is regulator independent owing to diffeomorphism invariance.

Whereas the investigation of homogeneous models in loop quantum cosmology will be more complicated than and very different from the standard treatment of minisuperspace quantizations, our constraint operators are, compared with the Hamiltonian constraint operator of the full theory, very similar but slightly simpler due to the symmetry. One simplification comes from the special nature of vertices being at most 6-valent. Also, the action is simpler because the graph of a spin network being acted on is not changed: no new edges are created but only the spins of existing ones and the vertex contractor are changed. Note that this implies that the large class of special solutions to the Hamiltonian constraint found in Ref. [12] has no counterpart in our cosmological models. Furthermore, the fact that the constraint algebra is represented anomaly free is realized rather trivially for the homogeneous models because the classical algebra is simpler, and so requiring an anomaly free representation is less restrictive as compared to the full theory [12, 25, 26]. The calculation of matrix elements of the constraints for Bianchi models can be done along the lines of Ref. [27], whereas the computation for LRS or isotropic models would be different because techniques for generalized spin networks are needed there.

Another difference to the full theory is that we do not have to face the issue of extracting

a smooth, semiclassical metric from our states. In the full theory, the quantum states represent a distributional metric with a discrete structure which, in order to perform a classical limit, have to be superposed in some way to semiclassical states. In a homogeneous regime, however, the metric is necessarily smooth and each state represents a homogeneous (but nevertheless quantum) metric. But we still have to understand how to interpret possible solutions of the constraint in a space-time picture or cosmological language, i.e. we have to interpret the dynamical constraint as an evolution equation [17, 18].

Acknowledgements

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